Infinite Horizon LQ

Given continuous-time state equation

\[ \dot{x} = Ax + Bu \]

Find the control function \( u(t) \) to minimize

\[ J = \frac{1}{2} \int_0^\infty \left[ x^T(t)Qx(t) + u^T(t)Ru(t) \right] dt \]

\( Q \geq 0, R > 0 \) and symmetric

Solution is obtained as the limiting case of Ricatti Eq.

Summary of Continuous-Time LQ Solution

Control law: \( u = -Kx, \quad K = R^{-1}B^TP \)

Algebraic Ricatti Equation (ARE)

\[ PA + A^TP + Q - PBR^{-1}B^TP = 0 \]

\[ J^* = \frac{1}{2} x(0)^T P x(0) \]
Example

Find the optimal controller that minimizes

\[
J = \frac{1}{2} \int_0^\infty \left[ y^2(t) + ru^2(t) \right] dt, \quad r > 0
\]

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u
\end{align*}
\]

\[
Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = r
\]

Solving for P

\[
PA + A^T P + Q - PBR^{-1} B^T P = 0
\]

Gives three simultaneous equations:

\[
\begin{align*}
1 - \frac{p_2^2}{r} &= 0 \\
p_1 - \frac{p_2 p_3}{r} &= 0 \\
2p_2 - \frac{p_3^2}{r} &= 0
\end{align*}
\]
Control Gain and Closed-Loop Poles

Control law: \( u = -Kx, \quad K = R^{-1} B^T P \)

\[
K = -\frac{1}{r} \begin{bmatrix} 0 & 1 \\ p_2 & p_3 \end{bmatrix} = -\frac{1}{r} \begin{bmatrix} p_2 & p_3 \end{bmatrix}
\]

\[
K = \begin{bmatrix} r^{-1/2} & \sqrt{2r}^{-1/4} \end{bmatrix}
\]

Closed-loop Characteristic Eq.

\[
|sI - A - BK| = s^2 + \sqrt{2r}^{-1/4}s + r^{-1/2} = 0
\]

Natural frequency and damping ratio

\[
\omega_n = r^{-1/4}, \quad \zeta = \frac{\sqrt{2}}{2} = 0.707
\]

Comments on the LQ Solution of Example

- As \( r \to 0 \), natural frequency \( \to \) infinity and settling time \( \to 0 \) (fast system)
- As \( r \to \infty \), natural frequency \( \to 0 \) and settling time \( \to \) infinity (slow system)
- Damping ratio is always 0.707
- Effect on \( u(t) \)?
**Optimal Estimation**

- Observer design based on pole-placement is nonunique and does not consider noise effects.
- Optimal observer dynamics are difficult to determine.
- Estimators can be optimized with respect to input and measurement noise properties.

**Discrete Random Variables**

Discrete random variable $x$ has discrete values $x_i$, $i=1,...,N$ each with certain occurrence probability $p_i$.

**Expected Value of $x$:**

$$\bar{x} = E(x) = \sum_{i=1}^{N} x_i p_i$$

**Covariance of $x$:**

$$P = E[(x - \bar{x})(x - \bar{x})^T] = \sum_{i=1}^{N} (x_i - \bar{x})(x_i - \bar{x})^T p_i$$
Dice Example

\[ x = \text{dots on rolled dice} \]

\[ \bar{x} = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \cdots + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36} = 7 \]

\[ P = (2 - 7)^2 \cdot \frac{1}{36} + \cdots + (12 - 7)^2 \cdot \frac{1}{36} = 5.83 \]

Continuous Random Variables

A continuous random variable \( x \) has continuous values from \(-\infty\) to \(+\infty\) according to certain probability density function (pdf) \( p(x) \):

Basic Properties of pdf:

\[ p(x) \geq 0 \]

\[ +\infty \]

\[ \int_{-\infty}^{+\infty} p(x) \, dx = 1 \]
Interpretation of pdf

\[ p(a) \, dx \text{ is the probability that } x \text{ lies in interval } \left[ a - \frac{dx}{2}, a + \frac{dx}{2} \right] \]

Expected value of \( x \):

\[ \bar{x} = E(x) = \int_{-\infty}^{+\infty} x p(x) \, dx \]

Covariance of \( x \):

\[ P = E[(x - \bar{x})^2] = \int_{-\infty}^{+\infty} (x - \bar{x})^2 p(x) \, dx \]

Uniform pdf Example

\[ p(x) = \begin{cases} \frac{1}{c}, & \frac{b - c}{2} \leq x \leq \frac{b - c}{2} \\ 0, & \text{otherwise} \end{cases} \]

\[ E(x) = \int_{-\infty}^{+\infty} x p(x) \, dx = \int_{-\infty}^{b+c/2} \frac{1}{c} x \, dx = b \]

\[ E[(x - b)^2] = \int_{b-c/2}^{b+c/2} \frac{1}{c} (x - b)^2 \, dx = \frac{c^2}{12} \]
Gaussian pdf Example

\[ p(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left[\frac{-\left(x - \bar{x}\right)^2}{2\sigma^2}\right] \]

It can be shown that

\[ E(x) = \bar{x} \]
\[ E\left[ (x - \bar{x})^2 \right] = \sigma^2 \]

Continuous Random Vectors

For \( x = [x_1 \, x_2 \ldots \, x_n] \), pdf \( p(x) \) is function of \( x_1, \ldots, x_n \):

Interpretation: \( p(a) \, dx_1, \ldots, dx_n \) is the probability that \( x \) lies in the differential volume \( dx_1, \ldots, dx_n \) centered at \( x = a \).

\[
\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} p(x) \, dx_1 \cdots dx_n = 1
\]
Expected Value and Covariance of a Random Vector

**Expected value of x:**

$$\bar{x} = E(x) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} xp(x) \, dx_1 \cdots dx_n$$

**Covariance of x:**

$$P = E[(x - \bar{x})(x - \bar{x})^T] = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} (x - \bar{x})(x - \bar{x})^T \, p(x) \, dx_1 \cdots dx_n$$

$$P = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \begin{bmatrix}
(x_1 - \bar{x}_1)^2 & \cdots & (x_1 - \bar{x}_1)(x_n - \bar{x}_n) \\
\vdots & \ddots & \vdots \\
(x_1 - \bar{x}_1)(x_n - \bar{x}_n) & \cdots & (x_n - \bar{x}_n)^2
\end{bmatrix} p(x) \, dx_1 \cdots dx_n$$

Covariance P is symmetric and semi positive definite

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Gaussian pdf

$$p(x) = \frac{1}{|P|^{1/2}(2\pi)^{n/2}} \exp\left[-\frac{1}{2} (x - \bar{x})^T P^{-1} (x - \bar{x})\right]$$

It can be shown that

$$E(x) = \bar{x}, \quad E[(x - \bar{x})(x - \bar{x})^T] = P$$
Maximum Likelihood Estimation (MLE)

Consider the static stochastic process:

\[ x \xrightarrow{A} w \xrightarrow{} y = Ax + w \]

\( x, w \): uncorrelated Gaussian random variables

**Problem:** Find the most likely estimate for \( x \) given observation \( y \)

**Expected Values and Covariances:**

\[
E[x] = \bar{x}, \quad E(w) = \bar{w}
\]

\[
E[(x - \bar{x})(x - \bar{x})^T] = P_{xx}, \quad E[(w - \bar{w})(w - \bar{w})^T] = P_{ww}
\]

\[
E[(x - \bar{x})(w - \bar{w})^T] = 0
\]

Least Squares Equivalent of MLE

**Joint pdf:**

\[
p(x, w) = \frac{1}{|P_{xx}|^{1/2} |P_{ww}|^{1/2} (2\pi)^n} \exp(-J)
\]

**LQ Index**

\[
J = \frac{1}{2} \tilde{x}^T P_{xx}^{-1} \tilde{x} + \frac{1}{2} \tilde{w}^T P_{ww}^{-1} \tilde{w}
\]

**Process Constraint:**

\[
\tilde{y} = A\tilde{x} + \tilde{w}
\]

where \( \tilde{y} = y - \bar{y}, \tilde{x} = x - \bar{x}, \tilde{w} = w - \bar{w} \)

MLE minimizes \( J \) subject to process constraint
MLE Solution

Minimize $J$

$$\frac{\partial J}{\partial \tilde{x}} = \tilde{x}^T P_{xx}^{-1} + \tilde{w}^T P_{ww}^{-1} \frac{\partial \tilde{w}}{\partial \tilde{x}} = 0, \quad \tilde{w} = \tilde{y} - A\tilde{x}$$

Solving for optimal $x$, denoted by $\hat{x}$, gives

$$\hat{x} = \bar{x} + \hat{P}_{xx} A^T P_{ww}^{-1} (y - A\bar{x} - \bar{w})$$

$$\hat{P}_{xx}^{-1} = P_{xx}^{-1} + A^T P_{ww}^{-1} A$$

Comments on MLE Solution

- Large $P_{ww}$ relative to $P_{xx}$ indicates lack of confidence in measurement $y$. This results in $x$ close to $\bar{x}$.

- The covariance of $x$ given $y$ (denoted $x|y$)

$$E[(x | y - \hat{x})(x | y - \hat{x})^T] = \hat{P}_{xx}$$

$$\hat{P}_{xx}^{-1} = P_{xx}^{-1} + A^T P_{ww}^{-1} A$$

This indicates a reduction in the initial uncertainty in $x$. 
Parameter Identification

Process: \( y_k = \varphi_k^T \theta + w_k \)

where

- \( y_k \): Output at time k (scalar)
- \( \theta \): Constant parameter vector (unknown)
- \( \varphi_k \): Regression vector (known)
- \( w_k \): Gaussian noise with zero mean

Find most likely estimate of \( \theta \) given \( y_0, y_1, \ldots, y_k \) denoted by \( \hat{\theta}_k \)

RLS Version of MLE

The MLE can be made recursive by using

- Current parameter estimate as the initial estimate in the MLE
- Current covariant estimate as the covariant of the initial estimate in the MLE

And continuing the process recursively
Recursive Least Squares

Exchanging the following variables in MLE

- $\bar{x} \rightarrow \hat{\theta}_{k-1}$, $\hat{x} \rightarrow \hat{\theta}_k$.
- $P_{xx} \rightarrow P_{k-1}$, $\hat{P}_{xx} \rightarrow P_k$, $P_{ww} \rightarrow 1$ (normalized)
- $\hat{\theta}_k = \hat{\theta}_{k-1} + P_k \varphi_k(\mathbf{y}_k - \varphi_k^T \hat{\theta}_{k-1})$
- $P_k^{-1} = P_{k-1}^{-1} + \varphi_k \varphi_k^T$

Efficient RLS Algorithm

- $\hat{\theta}_k = \hat{\theta}_{k-1} + \frac{P_{k-1} \varphi_k}{1 + \varphi_k^T P_{k-1} \varphi_k} (\mathbf{y}_k - \varphi_k^T \hat{\theta}_{k-1})$
- $P_k = P_{k-1} - \frac{P_{k-1} \varphi_k \varphi_k^T P_{k-1}}{1 + \varphi_k^T P_{k-1} \varphi_k}$
- $P_0 = E \left[ (\theta - \hat{\theta}_0)(\theta - \hat{\theta}_0)^T \right]$

- RLS changes $\hat{\theta}_k$ proportional to previous estimation error $(\mathbf{y}_k - \varphi_k^T \hat{\theta}_{k-1})$ and $P_{k-1} \varphi_k$. 
Parameter Convergence

- At every iteration \( P_k \leq P_{k-1} \), i.e., less uncertainty in parameter estimation
- If the regression vector is such that

\[
\lim_{k \to \infty} \lambda_{\min} \left( \sum_{k=0}^{\infty} \varphi_k \varphi_k^T \right) = \infty
\]

then \( \lim_{k \to \infty} P_k = 0 \) and \( \lim_{k \to \infty} \hat{\theta}_k = \theta \)

System Identification

Consider the Input-Output Model of a LTI Discrete system

\[
y(k) = \sum_{i=1}^{n} a_i y(k-i) + \sum_{i=1}^{n} b_i u(k-i) + w(k)
\]

Defining, \( y_k = y(k), w_k = w(k) \)

\[
\varphi_k = \begin{bmatrix} y(k-1) & \cdots & y(k-n) & u(k-1) & \cdots & u(k-n) \end{bmatrix}^T
\]

\[
\theta = \begin{bmatrix} a_1 & \cdots & a_n & b_1 & \cdots & b_n \end{bmatrix}
\]

Then \( y_k = \varphi_k^T \theta + w_k \)

RLS can be used to estimate unknown parameters
First Order Example

Consider the first order system

\[ y(k) = ay(k-1) + bu(k-1) + w(k) \]

Defining, \( y_k = y(k), w_k = w(k) \)

\[ \varphi_k = \begin{bmatrix} y(k-1) \\ u(k-1) \end{bmatrix}, \theta = \begin{bmatrix} a \\ b \end{bmatrix} \]

Then \( y_k = \varphi_k^T \theta + w_k \)

RLS can be used to estimate unknown parameters \( a \) and \( b \)

Matlab Solution: Input-Output Data

```matlab
% Parameter Identification Using RLS
% Continuous-Time plant \( G(s) = \frac{1}{s+1} \);
numc=[0 1]; denc=[1 1];
Ts=0.1;
[num,den]=c2dm(numc,denc,Ts,'zoh');
% Generate Input/Output Signal
K=[0:500]'; imax=length(K);
U=sin(2*pi*K/10);
Y=dlsim(num,den,U);
% Add Measurement Noise
ymax=max(abs(Y));
Noise=normrnd(0,0.5*ymax,size(Y));
Yn=Y+Noise;
```
Matlab Version of RLS

```matlab
%Initialize Least Squares Variables
phi=zeros(2,1); P=1000*eye(2,2);
thh=zeros(2,1); THH=[];

for ii=1:imax
    y=Yn(ii);
    [thhn,Pn]=rls(thh,P,y,phi);
    thh=thhn; P=Pn;
    THH=[THH; ii thh' y-phi'*thh]
    phi=[Yn(ii);U(ii)];
end
```

RLS Update Function

```matlab
function [thn,Pn]=rls(th,P,y,phi);
% This function updates the estimated
% parameter and the covariant matrix
% using the Recursive Least Squares
% (RLS) update algorithm.

e=y-phi'*th;
psi=P*phi;
g=1/(1+phi'*psi);
thn=th+g*psi*e;
Pn=(P-g*psi*psi');
```

Identification Results

![Identification Results Graph](image1)

Output With Noise

![Output With Noise Graph](image2)
Identification Results

![Graph showing Identification Results with axes for Amplitude vs. K]